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Exact Collision Detection of Two Moving Ellipsoids under Rational Motions

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***Abstract*— In this paper, we describe an exact method for detecting collision between two moving ellipsoids under pre-specified rational motions. Our method is based on an algebraic condition that determines the separation status of two static ellipsoids – the condition itself is described by the signs of roots of the characteristic equation of the two ellipsoids. To deal with moving ellipsoids, we derive a bivariate function whose zero-set possesses a special topological structure. By analyzing the zero-set of this function, we are able to tell whether or not two moving ellipsoids under pre-specified rational motions are collision-free; and if not, we can determine the intervals in which they overlap.**

I. Introduction

Designing and analyzing motions of objects in 3D space are of importance to robotics, computer graphics, 3D computer games [4] and CAD applications. To simulate a dynamic environment, it is critical to ensure that rigid objects do not penetrate each other and impulsive response is properly handled when objects collide. Collision detection is to determine whether or not two moving objects come into contact with each other.

To perform direct collision detection for general objects is computationally complicated. Using bounding volumes to enclose complex shaped objects reduces the computational cost by first detecting collision on the bounding volumes. Ellipsoids are often chosen as bounding volumes for robotic arms in collision detection because of their flexibilities in shape [8], [14], [20]. Enclosing and enclosed ellipsoids are studied for collision avoidance between convex polyhedra in [8], [16] using heuristic solutions. Rimon and Boyd [14] present an efficient numerical technique for computing the *quasi-distance*, called *margin*, between two separate ellipsoids using an incremental approach. In [17], Sohn et al. compute the distance between two ellipsoids using line geometry. Using the Lagrange conditions, Lennerz and Schömer [11] present an efficient algorithm for computing the distance for quadratic curves and surfaces. Ellipsoids are also used to represent soil particles in geo-mechanics and the iso-potential surface of a molecule in computational physics. The problem of determining whether or not two ellipsoids overlap is also studied in these fields [12], [13]. However, the proposed solutions are also numerical techniques for static ellipsoids, and cannot be applied to exact (i.e., non-incremental) collision detection between two moving ellipsoids following pre-specified closed-form motion paths.

To deal with moving ellipsoids, a brute-force approach is to perform interference testing between two static ellipsoids along the motion path at discrete time intervals. However, errors may be incurred due to inadequate temporal sampling resolution. In this paper, we present an exact collision detection method for two ellipsoids moving under pre-specified rational motions. We make use of an algebraic condition given in [19] that decides whether or not two static ellipsoids are separate by root characterization of the characteristic equation of the two ellipsoids. We extend the characteristic equation to take into account the time parameter and then perform collision detection using the zero-set of the bivariate characteristic equation of the two moving ellipsoids.

We first present in Section II the preliminaries for formulating the collision detection problem of two moving ellipsoids under rational motions. Our collision detection solution is then described in details in Section III. A numerical example is provided to illustrate how our method works. We then give some experimental results in Section IV and finally the paper is concluded in Section V where we outline some open problems and future research directions.

II. Preliminaries

Throughout this paper, we assume that an ellipsoid $\mathscr{A}$ is given by a quadratic equation $X^TAX = 0$ in $\mathbb{E}^3$, where $X = (x,y,z,w)^T$ are the homogeneous coordinates of a point in 3D space. The symmetric coefficient matrix A is normalized so that the interior of $\mathscr{A}$ is given by $X^TAX < 0$. We also use $\mathscr{A}$ to denote the closed point set comprising of the boundary points and interior points of the ellipsoid, and use $\mathsf{Int}(\mathscr{A})$ to denote the interior points of $\mathscr{A}$. Given two ellipsoids $\mathscr{A} : X^TAX = 0$ and $\mathscr{B} : X^TBX = 0$, they are said to be *separate* or *disjoint* if $\mathscr{A} \bigcap \mathscr{B} = \emptyset$, and *overlapping* if $\mathscr{A} \bigcap \mathscr{B} \neq \emptyset$. Then $\mathscr{A}$ and $\mathscr{B}$ are said to be *touching externally* if $\mathscr{A} \bigcap \mathscr{B} \neq \emptyset$ and $\mathsf{Int}(\mathscr{A}) \bigcap \mathsf{Int}(\mathscr{B}) = \emptyset$.

In this section, we first give the separation condition for two ellipsoids in $\mathbb{E}^3$ which is rigorously proved in [19]. Then, we introduce 3D rational motions that are the assumed motions taken by two moving ellipsoids in this study.

A. Algebraic condition for the separation of two ellipsoids

For two ellipsoids $\mathscr{A} : X^T AX = 0$ and $\mathscr{B} : X^T BX = 0$ in $\mathbb{E}^3$, the quartic polynomial $\mathrm{f}(\lambda) = \det(\lambda \mathrm{A} - \mathrm{B})$ is called the *characteristic polynomial* and $\mathrm{f}(\lambda) = 0$ is called the *characteristic equation* of $\mathscr{A}$ and $\mathscr{B}$. Fig. 1 shows two ellipsoids and their corresponding characteristic polynomial $\mathrm{f}(\lambda)$. The relationship between the geometric configuration of two ellipsoids and the roots of their characteristic equation is studied in [19] and is stated as follows:

Theorem 1 (Separation Condition of Two Static Ellipsoids in $\mathbb{E}^3$) *Let $\mathscr{A}$ and $\mathscr{B}$ be two ellipsoids with characteristic equation $\mathrm{f}(\lambda) = 0$. Then,*

1) *$\mathscr{A}$ and $\mathscr{B}$ are separate if and only if $\mathrm{f}(\lambda) = 0$ has two distinct negative roots;*
2) *$\mathscr{A}$ and $\mathscr{B}$ touch each other externally if and only if $\mathrm{f}(\lambda) = 0$ has a negative double root.*

Remark: Note that the theorem given in [19] assumes that the characteristic equation be given in the form of $\mathrm{f}(\lambda) = \det(\lambda A + B) = 0$ and therefore the result there is stated in terms of positive roots. We make the changes here in consistency with the classic literature in linear algebra.

The above theorem enables us to determine whether or not two ellipsoids are separate by considering only the existence of distinct negative roots, without the need of solving the quartic equation. This can be done by using the method of Sturm's sequence [3] that computes the number of real zeros of a polynomial of any degree in a given interval.

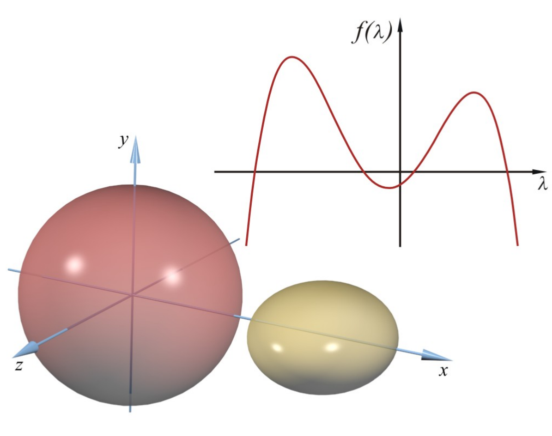


Fig. 1. Two ellipsoids and their characteristic polynomial $\mathrm{f}(\lambda)$.

B. 3D rational Euclidean motions

A rational Euclidean motion in $\mathbb{E}^3$ is given by

$$M(t) = \begin{pmatrix} R(t) & V(t) \\ \mathbf{0^T} & 1 \end{pmatrix}, \tag{1}$$

where $V(t) \in \mathbb{E}^3$, $R(t)$ is a 3×3 orthogonal matrix, and t can be considered as a parameter of time. The motion is a composition of a rotation $R(t)$ acting upon a point in $\mathbb{E}^3$, followed by a translation $V(t)$. All rational motions can be represented in (1) with

$$V(t) = \left(\frac{v_0}{v_3}, \frac{v_1}{v_3}, \frac{v_2}{v_3} \right)^T$$

and

$$R(t) = \frac{1}{E} \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2e_1e_2 - 2e_0e_3 & 2e_0e_2 + 2e_1e_3 \\ 2e_0e_3 + 2e_1e_2 & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2e_2e_3 - 2e_0e_1 \\ 2e_1e_3 - 2e_0e_2 & 2e_0e_1 + 2e_2e_3 & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix} \tag{2}$$

where $E = e_0^2 + e_1^2 + e_2^2 + e_3^2$ and $v_0, \ldots, v_3, e_0, \ldots, e_3$ are polynomials in t [10]. Note that e_0, e_1, e_2, e_3 are the Euler parameters that describe a rotation about a vector in $\mathbb{E}^3$. We call them *normalized Euler parameters* when $E = 1$. Readers are referred to [15] for a survey on rational motion design and [6], [9], [10] for interpolating a set of positions in $\mathbb{E}^3$ using piecewise B-spline motions.

When the entries of $V(t)$ and $R(t)$ are polynomials of maximal degree k, we called $M(t)$ a *rational motion of degree k*. A moving ellipsoid $\mathscr{A}(t)$ undertaking a rational motion $M(t)$ is represented as $X^T A(t)X = 0$, where

$$A(t) = (M^{-1}(t))^T AM^{-1}(t).$$

Assume that the maximal degree of the entries in $R(t)$ and $V(t)$ are k_R and k_V, respectively. Then, $M^{-1}(t)$ and $A(t)$ are given by

$$M^{-1}(t) = \begin{pmatrix} R(t)^T{}_{<k_R>} & -R(t)^T V(t)_{<k_R+k_V>} \\ \mathbf{0^T} & 1 \end{pmatrix},$$

and

$$A(t) = \begin{pmatrix} P(t)_{<2k_R>} & U(t)_{<2k_R+k_V>} \\ U(t)^T{}_{<2k_R+k_V>} & s(t)_{<2(k_R+k_V)>} \end{pmatrix} \tag{3}$$

for some 3×3 matrix $P(t)$, 3-vector $U(t)$, and scalar function $s(t)$. Here, the bracketed subscript represents the maximal degree of the entries of the associated entity.

III. Collision Detection of Two Moving Ellipsoids

In this section, we present a collision detection scheme for determining whether or not two moving ellipsoids under rational motions are collision-free.

Given two moving ellipsoids $\mathscr{A}(t) : X^T A(t)X = 0$ and $\mathscr{B}(t) : X^T B(t)X = 0$ under rational motions $M_A(t)$ and $M_B(t)$, $t \in [0,1]$, respectively, $\mathscr{A}(t)$ and $\mathscr{B}(t)$ are said to be *collision-free* if $\mathscr{A}(t)$ and $\mathscr{B}(t)$ are separate for all $t \in [0,1]$; otherwise, $\mathscr{A}(t)$ and $\mathscr{B}(t)$ *collide*.

The characteristic equation of $\mathscr{A}(t)$ and $\mathscr{B}(t)$, $t \in [0,1]$, is then given by

$$\mathrm{f}(\lambda;t) = \det(\lambda A(t) - B(t)) = 0.$$

At any time $t_0 \in [0,1]$, if $\mathscr{A}(t_0)$ and $\mathscr{B}(t_0)$ are separate, then $\mathrm{f}(\lambda;t_0) = 0$ has two distinct negative roots; otherwise, $\mathscr{A}(t_0)$ and $\mathscr{B}(t_0)$ are either touching externally or overlapping, and $\mathrm{f}(\lambda;t_0) = 0$ has a double negative root or no negative root, respectively. Hence, we need to generate the zero-set of $\mathrm{f}(\lambda;t) = 0$, i.e., all the points (λ,t) that satisfy the equation $\mathrm{f}(\lambda;t) = 0$. Write

$$\mathrm{f}(\lambda;t) = g_4(t)\lambda^4 + g_3(t)\lambda^3 + g_2(t)\lambda^2 + g_1(t)\lambda + g_0(t).$$

Suppose the maximum degrees of entries in $R_A(t)$ and $V_A(t)$ of the motion $M_A(t)$ are k_R and k_V, respectively; and the maximum degrees of entries in $R_B(t)$ and $V_B(t)$ of the motion $M_B(t)$ are also k_R and k_V, respectively. Then each entry in $\lambda A(t) - B(t)$ will have the same degree as the corresponding entry in $A(t)$ given by (3), and the degree of the coefficients $g_i(t), i = 0,\ldots,4$, is therefore

$$k_f = 3(2k_R) + 2(k_R + k_V) = 8k_R + 2k_V.$$

Notice that the $g_i(t), i = 0,\ldots,4$, are continuous functions in t and as described in [1], the four roots of $\mathrm{f}(\lambda;t) = 0$ are also continuous functions in t. Therefore, the zero-set of $\mathrm{f}(\lambda;t) = 0$ is a collection of curve segments or loops in the λ-t plane. Let $\mathscr{Z}$ denote such a collection containing only segments or loops with negative λ .

Since $\mathrm{f}(\lambda;t) = 0$ has special root patterns, $\mathscr{Z}$ has special topological structure, i.e., there can only be either 2 distinct, 1 double or no negative λ for any t; or, geometrically, a line $t = t_0$ intersects the zero-set of $\mathrm{f}(\lambda;\mathrm{t}) = 0$ in at most two points with $\lambda < 0$. Fig. 2 shows four typical layout of the curve $\mathscr{Z}$, assuming that the two ellipsoids are separate initially. Now the curve $\mathscr{Z}$ always starts with two separate branches at $t = 0$. These two branches remain separate if the two moving ellipsoids are collision-free, as shown in Fig. 2(a). These two branches merge into one point at some $t = t_0$, i.e., there is a double negative root for $\mathrm{f}(\lambda;t_0) = 0$ (see Fig. 2(b)), if the two moving ellipsoids make contact only at time $t = t_0$. Finally, if there is no curve segment in the strip $(-\infty,0) \times (t_0,t_1)$, then there is no negative root within the interval (t_0,t_1), so $\mathscr{A}(t)$ and $\mathscr{B}(t)$ are overlapping in the time interval (t_0,t_1) (see Fig. 2(c) & (d)). Hence, we can determine whether two given moving ellipsoids are collision-free or not; and if they collide, we may also report all collision intervals through an analysis of the topology of the zero-set $\mathscr{Z}$.

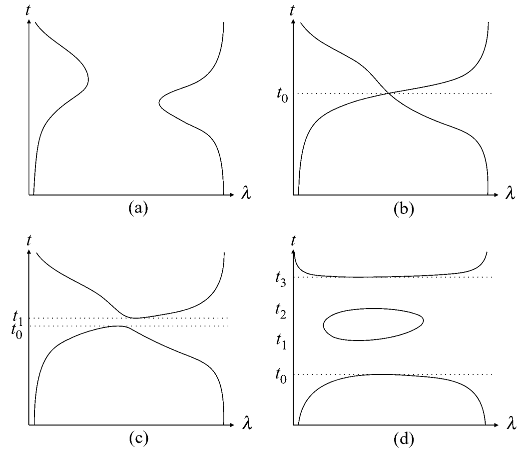


Fig. 2. Typical topologies of the zero sets for two moving ellipsoids that are (a) separate because there are two distinct negative roots for all t; (b) separate until $t = t_0$ when they touch each other and then separate again; (c) overlapping when $t \in [t_0,t_1]$ indicated by the absence of negative roots; (d) overlapping when $t \in [t_0,t_1]$ and $[t_2,t_3]$.

In order to extract the zero-set of $\mathrm{f}(\lambda;t) = 0$ in a robust manner, we first convert $\mathrm{f}(\lambda;t) = 0$ into a Bernstein representation by the variable substitution

$$\lambda = \frac{-u}{1-u}, \quad \text{and} \quad t = \frac{v}{1-v},$$

which maps the domain $(\lambda,t) \in (-\infty,0] \times [0,1]$ to $(u,v) \in [0,1] \times [0,\frac{1}{2}]$, and accordingly,

$$\begin{aligned}
\mathrm{f}(\lambda;t) &= \sum_{i=0}^{4}\sum_{j=0}^{k_f} a_{ij}\lambda^i t^j \\
&= \sum_{i=0}^{4}\sum_{j=0}^{k_f} a_{ij}\left(\frac{-u}{1-u}\right)^i\left(\frac{v}{1-v}\right)^j \\
&= \frac{1}{d}\sum_{i=0}^{4}\sum_{j=0}^{k_f} a_{ij}(-1)^i u^i(1-u)^{4-i}v^j(1-v)^{k_f-j} \\
&= \frac{1}{d}\sum_{i=0}^{4}\sum_{j=0}^{k_f} b_{ij}B_{4,i}(u)B_{k_f,j}(v) \\
&\equiv \frac{1}{d}S(u,v),
\end{aligned}$$

where

$$a_{ij} \in \mathbb{R}, \; b_{ij} = \frac{(-1)^i}{\binom{4}{i}\binom{k_f}{j}}a_{ij}, \; d = (1-u)^4(1-v)^{k_f},$$

and $B_{m,n}(u) = \binom{m}{n} u^n (1-u)^{m-n}$ is the Bernstein basis function, and k_f is the degree of the coefficients of $\mathrm{f}(\lambda;\mathrm{t})$ in t.

Let $\mathscr{S}$ denote the zero-set of $S(u,v)$. Clearly, the zero-set of $\mathrm{f}(\lambda;t)$ and the zero-set of $S(u,v)$ have the same topology. For the extraction of the zero-set of $S(u,v)$, we used the IRIT system, a geometric modeling package developed at Technion, Israel. (See Elber and Kim [5] for details of the implemented algorithm.) The zero-set $\mathscr{S}$ was computed by IRIT and organized into connected components. By projecting these components to the t-axis, we can determine all the intervals of t, if any, over which $\mathscr{S}$ does not have a curve segment. This enables us to determine whether or not the two ellipsoids are collision-free, and report all the collision intervals when they are not.

A Numerical Example

Consider two ellipsoids $\mathscr{A}: \frac{x^2}{5} + \frac{y^2}{8} + \frac{z^2}{10} = 1$ and $\mathscr{B}: \frac{x^2}{10} + \frac{y^2}{5} + \frac{z^2}{4} = 1$ moving under rational motions (Fig. 3(a)) given by

$$M_A(t) = \begin{pmatrix} R_A(t) & V_A(t) \\ \mathbf{0}^\mathbf{T} & 1 \end{pmatrix} \quad \text{and}$$

$$M_B(t) = \begin{pmatrix} R_B(t) & V_B(t) \\ \mathbf{0}^\mathbf{T} & 1 \end{pmatrix},$$

with

$$V_A(t) = \begin{pmatrix} 17t^2 + 66t - 63 \\ -138t^2 + 150t - 35 \\ -43t^2 + 34t - 5 \end{pmatrix}, V_B(t) = \begin{pmatrix} 8t^2 + 80t - 70 \\ 126t^2 - 120t + 10 \\ -32t^2 + 54t - 20 \end{pmatrix}.$$

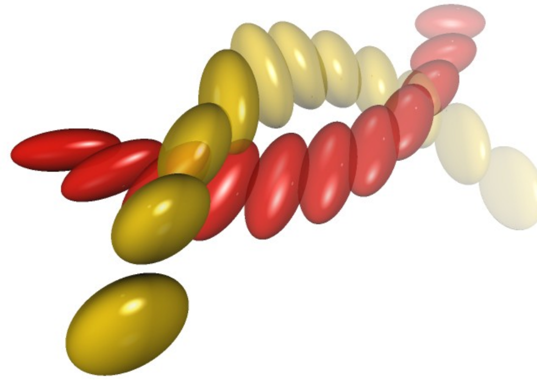
The rotational matrices $R_A(t)$ and $R_B(t)$ take the form as in (2) with Euler parameters $\{e_0^{(A)}, e_1^{(A)}, e_2^{(A)}, e_3^{(A)}\}$ and $\{e_0^{(B)}, e_1^{(B)}, e_2^{(B)}, e_3^{(B)}\}$, respectively, where

$$\begin{cases} e_0^{(A)} = 0 & e_1^{(A)} = 1.8t^2 - 2.4t + 0.6 \\ e_2^{(A)} = -1.6t + 0.8 & e_3^{(A)} = -t^2 + 1.6t \end{cases}$$

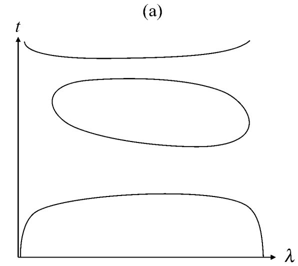
and

$$\begin{cases} e_0^{(B)} = 0 & e_1^{(B)} = 1.6t^2 - 1.6t \\ e_2^{(B)} = 0.4t^2 + 0.6 & e_3^{(B)} = 0.8t^2 - 1.6t + 0.8 \end{cases}.$$

Since the zero-set of their characteristic equation $\mathrm{f}(\lambda;\mathrm{t}) = 0$ (as shown in Fig. 3(b)) contains curve segments in the intervals $[0, 0.1699]$, $[0.3465, 0.7055]$ and $[0.8544, 1]$ (corr. to 4 sig. digits), we conclude that the two moving ellipsoids collide and, in particular, they are overlapping or touching within the time intervals $[0.1699, 0.3465]$ and $[0.7055, 0.8544]$.



(a)



(b)

Fig. 3. Two ellipsoids moving under quartic rational motions. (a) The motion paths; and (b) the corresponding zero-set of their characteristic equation.

IV. Experimental Results

We have tested our collision detection method on ellipsoids moving under rational motions of different degrees. The computations are carried out in double precision on a Pentium 4 2GHz CPU. The results are shown in Table I. Higher order motion leads to an increase in the degree in t of the coefficients of the characteristic polynomial $\mathrm{f}(\lambda;\mathrm{t})$, which in turn results in a longer execution time. For quartic motions, a single collision detection takes about 4 seconds. It is worth noting that a considerable amount of time has been spent on the zero-set extraction module which computes the intersection of a Bézier surface and a plane, as compared to the time used in obtaining the characteristic polynomial. Therefore the use of an efficient geometric intersection scheme is critical to the overall performance of our collision detection method. It is also found that the time taken for computing the zero-set for collision-free cases is in general longer than that for colliding cases, because of the existence of longer intersection curves in the zero-set when two moving ellipsoids do not collide.

As have been mentioned before, the degree of the coefficients of $\mathrm{f}(\lambda;\mathrm{t})$ can be up to $k_f = 8k_R + 2k_V$ in t. In fact, when we apply our collision detection method to ellipsoids moving under degree 6 rational motions, numerical instability becomes a major problem for ob-

TABLE I

AVERAGE CPU TIME TAKEN FOR RATIONAL MOTIONS OF VARIOUS DEGREES. THE RATIONAL MOTION $M(t)$ IS COMPOSED OF $R(t)$ AND $V(t)$ GIVEN BY (1). THE COLUMNS (I) AND (II) SHOW THE TIME TAKEN IN POLYNOMIAL SET-UP AND ZERO-SET EXTRACTION, RESPECTIVELY.

Degree in t				Average CPU time (sec)		
$R(t)$	$V(t)$	$M(t)$	$\mathrm{f}(\lambda;t)$	(I)	(II)	overall
0	1	1	2	0.005	0.415	0.420
2	1	2	18	0.008	1.104	1.113
4	2	4	36	0.009	4.739	4.748

taining an accurate answer. Errors come from both the evaluation of the high degree characteristic polynomial and the extraction of the zero-set. The possibility of tackling this limitation for high degree motions by using multi-precision computations remains to be explored.

We have also applied a sampling technique to approximate the zero-set of the bivariate characteristic equation $\mathrm{f}(\lambda;t) = 0$. Using a marching square technique applied to a regular grid of 100×100 cells, we have experienced a considerable speed-up of at least 250 times in computing the zero-set of $\mathrm{f}(\lambda;t) = 0$. This sampling-based approach is not exact; and there is some danger of missing small loops in the zero-set. However, in many practical situations, this approximate technique would be quite useful. Almost all contemporary algorithms for collision-detection are based on certain sampling techniques. In the future work, we shall investigate various ways of speeding-up the zero-set computation, which include methods for approximation as well as for exact computation of the zero-set.

V. CONCLUSIONS

We have presented an exact collision detection method for two ellipsoids moving under rational motions. The method is based on an algebraic separation condition for two static ellipsoids. Although the separation condition involves the characterization of the roots of the characteristic equation of two ellipsoids, there is in fact no need to solve for the roots of the quartic equation. Our collision detection method is based on this algebraic condition and uses the characteristic equation $\mathrm{f}(\lambda;\mathrm{t}) = 0$ of two moving ellipsoids. By extracting the zero-set of this bivariate function, we can easily determine whether or not two moving ellipsoids are collision-free during their motions.

Although we consider only Euclidean rational motions in this study, our collision detection method can in fact be applied without any modifications to ellipsoids moving with affine spline motions [7]. These ellipsoids may also change their shapes during motion and therefore are very suitable for applications such as computer animations and morphing.

VI. ACKNOWLEDGMENTS

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